A new Kim's type Bernoulli and Euler Numbers and related identities and zeta and L-functions

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Abstract. In this paper, by using q-deformed bosonic p-adic integral, we give λ-Bernoulli numbers and polynomials, we prove Witt's type formula of λ-Bernoulli polynomials and Gauss multiplicative formula for λ-Bernoulli polynomials. By using derivative operator to the generating functions of λ-Bernoulli polynomials and generalized λ-Bernoulli numbers, we give Hurwitz type λ-zeta functions and Dirichlet's type λ-L-functions; which are interpolated λ-Bernoulli polynomials and generalized λ-Bernoulli numbers, respectively. We give generating function of λ-Bernoulli numbers with order r. By using Mellin transforms to their function, we prove relations between multiply zeta function and λ-Bernoulli polynomials and ordinary Bernoulli numbers of order r and λ-Bernoulli numbers, respectively. We also study on λ-Bernoulli numbers and polynomials in the space of locally constant. Moreover, we define λ-partial zeta function and interpolation function.

§0. Introduction, definitions and notations

Throughout this paper, $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will be denoted by the ring of rational integers, the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$, cf. [2,3,4,5,6,7,8,9,16,17, 21,27].

When one talks of q−extension, q considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, as p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the following notations:

$$[x] = [x : q] = \frac{1 - q^x}{1 - q}, \quad \text{cf } [3, 4, 5, 6, 8, 9, 24, 25, 27].$$

Observe that when $\lim_{q \to 1}[x] = x$, for any x with $|x|_p \leq 1$ in the present p-adic case $[x : a] = \frac{1 - a^x}{1 - a}$.

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Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$\mathbb{X} = \varprojlim_N \left(\mathbb{Z}/dp^N \mathbb{Z} \right),$$

$$\mathbb{X}^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N \mathbb{Z}_p = \{ x \in \mathbb{X} | x \equiv a \ (\mathrm{mod}\ dp^n) \},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. We assume that $u \in \mathbb{C}_p$ with $|1 - u|_p \ge 1$. cf. [3,4,5,6,7,8,24, 27].

For $x \in \mathbb{Z}_p$, we say that g is a uniformly differentiable function at point $a \in \mathbb{Z}_p$, and write $g \in UD(\mathbb{Z}_P)$, the set of uniformly differentiable functions, if the difference quotients,

$$F_g(x, y) = \frac{g(y) - g(x)}{y - x},$$

have a limit $l = g'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, the *q-deformed bosonic p-adic integral* was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x)$$

$$= \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) \mu_q(x + p^N \mathbb{Z}_p) \qquad (A)$$

$$= \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) \frac{q^x}{[p^N]}, \quad [4,5,9].$$

By Eq-(A), we have

$$\lim_{q \to -q} I_q(f) = I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x).$$

This integral, $I_{-q}(f)$, give the q-deformed integral expression of fermioinc. The classical Euler numbers were defined by means of the following generating function:

$$\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}, \quad |t| < 2\pi. \quad [6, 7, 19, 21].$$

Let u be algebraic in complex number field. Then Frobenius-Euler polynomials [6,7,19,21] were defined by

$$\frac{1 - u}{e^t - u} e^{xt} = e^{H(u,x)t} = \sum_{m=0}^{\infty} H_m(u, x) \frac{t^m}{m!}, \qquad (A1)$$

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where we use technical method's notation by replacing $H^m(u, x)$ by $H_m(u, x)$ symbolically. In case $x = 0$, $H_m(u, 0) = H_m(u)$, which is called Frobenius-Euler number. The Frobenius-Euler polynomials of order r, denoted by $H_n^{(r)}(u, x)$, were defined by

$$\left(\frac{1-u}{e^t - u}\right)^r e^{tx} = \sum_{n=0}^{\infty} H_n^{(r)}(u, x)\frac{t^n}{n!} \quad \text{cf.}[7, 10, 26, 27].$$

The values at $x = 0$ are called Frobenius-Euler numbers of order r. When $r = 1$, these numbers and polynomials are reduced to ordinary Frobenius-Euler numbers and polynomials. In the usual notation, the n-th Bernoulli polynomial were defined by means of the following generating function:

$$\left(\frac{t}{e^t - 1}\right) e^{tx} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

For $x = 0$, $B_n(0) = B_n$ are said to be the n-th Bernoulli numbers. The Bernoulli polynomials of order r were defined by

$$\left(\frac{t}{e^t - 1}\right)^r e^{tx} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}$$

and $B_n^{(r)}(0) = B_n^{(r)}$ are called the Bernoulli numbers or order r. Let $x, w_1, w_2, \cdots, w_r$ be complex numbers with positive real parts. When the generalized Bernoulli numbers and polynomials were defined by means of the following generating function:

$$\frac{w_1 w_2 \cdots w_r t^r e^{xt}}{(e^{w_1 t} - 1)(e^{w_2 t} - 1) \cdots (e^{w_r t} - 1)} = \sum_{n=0}^{\infty} B_n^{(r)}(x \mid w_1, w_2 \cdots, w_r)\frac{t^n}{n!}$$

and $B_n^{(r)}(0 \mid w_1, w_2 \cdots, w_r) = B_n^{(r)}(w_1, w_2 \cdots, w_r)$, cf. [13,15].

The Hurwitz zeta function is defined by

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(x + n)^s},$$

$\zeta(s, 1) = \zeta(s)$, which is Riemann zeta function. The multiple zeta functions [12,27] were defined by

$$\zeta_r(s) = \sum_{0 < n_1 < n_2 < \cdots < n_r} \frac{1}{(n_1 + \cdots + n_r)^s}. \tag{C}$$

We summarize our paper as follows:

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In section 1, by using q-deformed bosonic p-adic integral, generating function of λ-Bernoulli numbers and polynomials are given. We obtain many new identities related to these numbers and polynomials. We proved Gauss multiplicative formula for λ-Bernoulli numbers. Witt's type formula of λ-Bernoulli polynomials is given.

In section 2, by using $\left. \left(\dfrac{d}{dt} \right)^k \right|_{t=0}$ derivative operator to the generating function of the λ-Bernoulli numbers, we define new relations and Hurwitz' type λ-function, which interpolates λ-Bernoulli polynomials at negative integers.

In section 3, by using same method of section 2, we give Dirichlet type λ-L-function which interpolates generalized λ-Bernoulli numbers.

In section 4, generating function of λ-Bernoulli numbers of order r is defined, by using Cauchy residue theorem and Mellin transforms to this function, we proved relation between multiple zeta function and λ-Bernoulli numbers of order r.

In section 5, we give some important identities related to generalized λ-Bernoulli numbers of order r.

In section 6, we study on λ-Bernoulli numbers and polynomials in the space of locally constant. In this section, we also define λ-partial zeta function which interpolates λ-Bernoulli numbers at negative integers.

In section 7, we give p-adic interpolation functions.

§1. λ-Bernoulli numbers

In this section, by using Eq-(A), we give integral equation of bosonic p-adic integral. By using this integral equation we define generating function of λ-Bernoulli polynomials. We give fundamental properties of the λ-Bernoulli numbers and polynomials. We also give some new identities related to λ-Bernoulli numbers and polynomials. We prove Gauss multiplicative formula for λ-Bernoulli numbers as well. Witt's type formula of λ-Bernoulli polynomials is given.

To give the expression of bosonic p-adic integral in Eq-(A), we consider the limit

$$I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x). \tag{0}$$

Bosonic p-adic integral on $\mathbb{Z}_p (= p$-adic invariant integral on $\mathbb{Z}_p)$

$$I_1(f_1) = I_1(f) + f'(0), \tag{1}$$

where $f_1(x) = f(x+1)$, integral equation for bosonic p-adic integral. Let C_{p^n} be the space of primitive p^n-th root of unity,

$$C_{p^n} = \{\zeta \mid \zeta^{p^n} = 1\}.$$

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Then, we denote
$$T_p = \lim_{n \to \infty} C_{p^n} = \bigcup_{n \geq 0} C_{p^n}.$$

For $\lambda \in \mathbb{Z}_p$, we take $f(x) = \lambda^x e^{tx}$, and $f_1(x) = e^t \lambda f(x)$. Thus we have

$$f_1(x) - f(x) = (\lambda e^t - 1)f(x). \tag{2}$$

By substituting (2) into (1), we get

$$(\lambda e^t - 1)I_1(f) = f'(0). \tag{2a}$$

Consequently, we have

$$\frac{\log \lambda + t}{\lambda e^t - 1} := \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}. \tag{3}$$

By using Eq-(3), we obtain

$$\lambda(B(\lambda) + 1)^n - B_n(\lambda) = \begin{cases} \log \lambda, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention of replacing $B_n(\lambda)$ by $B^n(\lambda)$. We give some $B_n(\lambda)$ numbers as follows:

$$B_0(\lambda) = \frac{\log \lambda}{\lambda - 1}, \quad B_1(\lambda) = \frac{\lambda - 1 - \lambda \log \lambda}{(\lambda - 1)^2}, \cdots .$$

We note that, if $\lambda \in T_p$, for some $n \in \mathbb{N}$, then Eq-(2a) is reduced to the following generating function:

$$\frac{t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}. \tag{3a}$$

If $\lambda = e^{2\pi i/f}$, $f \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, then Eq-(3) is reduced to (3a). Eq-(3a) is obtained by Kim [3]. Let $u \in \mathbb{C}$, then by substituting $x = 0$ into Eq-(A1), we set

$$\frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}. \tag{3b}$$

$H_n(u)$ is denoted Frobenius-Euler numbers. Relation between $H_n(u)$ and $B_n(\lambda)$ is given by the following theorem:

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Theorem 1. *Let $\lambda \in \mathbb{Z}_p$. Then*

$$B_n(\lambda) = \frac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \frac{nH_{n-1}(\lambda^{-1})}{\lambda - 1}, \tag{4}$$

$$B_0(\lambda) = \frac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1}).$$

Proof. By using Eq-(3), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!} &= \frac{\log \lambda + t}{\lambda e^t - 1} = \frac{\log \lambda}{\lambda e^t - 1} + \frac{t}{\lambda e^t - 1} \\
&= \frac{1 - \lambda^{-1}}{(1 - \lambda^{-1})\lambda} \cdot \left(\frac{\log \lambda}{e^t - \lambda^{-1}} \right) - \frac{(1 - \lambda^{-1})}{(e^t - \lambda^{-1})} \cdot \frac{t}{\lambda(1 - \lambda^{-1})} \\
&= \frac{\log \lambda}{\lambda - 1} \sum_{n=0}^{\infty} H_n(\lambda^{-1}) \frac{t^n}{n!} + \frac{t}{\lambda - 1} \sum_{n=0}^{\infty} H_n(\lambda^{-1}) \frac{t^n}{n!},
\end{aligned}$$

the next to the last step being a consequence of Eq-(3b). After some elementary calculations, we have

$$\sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!} = \frac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1}) + \sum_{n=1}^{\infty} \left(\frac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \frac{n}{\lambda - 1} H_{n-1}(\lambda^{-1}) \right) \frac{t^n}{n!}.$$

By comparing coefficient $\dfrac{t^n}{n!}$ in the above, then we obtain the desired result. $\square$

Observe that, if $\lambda \in T_p$ in Eq-(4), then we have, $B_0(\lambda) = 0$ and $B_n(\lambda) = \dfrac{nH_{n-1}(\lambda^{-1})}{\lambda - 1}, n \geq 1$.

By Eq-(3) and Eq-(4), we obtain the following formula:

For $n \geq 0, \lambda \in \mathbb{Z}_p$

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = \begin{cases} \dfrac{\log \lambda}{\lambda - 1} H_0(\lambda^{-1}), & n = 0 \\[3mm] \dfrac{\log \lambda}{\lambda - 1} H_n(\lambda^{-1}) + \dfrac{n}{\lambda - 1} H_{n-1}(\lambda^{-1}), & n > 0 \end{cases} \tag{4a}$$

and

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = B_n(\lambda), \quad n \geq 0. \tag{4b}$$

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Now, we define $\lambda-$Bernoulli polynomials, we use these polynomials to give the sums powers of consecutive. The λ-Bernoulli polynomials are defined by means of the following generating function:

$$\frac{\log \lambda + t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(\lambda; x)\frac{t^n}{n!}. \tag{5}$$

By Eq-(3) and Eq-(5), we have

$$B_n(\lambda; x) = \sum_{k=0}^{n} \binom{n}{k} B_k(\lambda)x^{n-k}.$$

The Witt's formula for $B_n(\lambda; x)$ is given by the following theorem:

Theorem 2. *For $k \in \mathbb{N}$ and $\lambda \in \mathbb{Z}_p$, we have*

$$B_n(\lambda; x) = \int_{\mathbb{Z}_p} (x + y)^n \lambda^y d\mu_1(y). \tag{6}$$

Proof. By substituting $f(y) = e^{t(x+y)}\lambda^y$ into Eq-(1), we have

$$\int_{\mathbb{Z}_p} e^{t(x+y)}\lambda^y d\mu_1(y) = \sum_{n=0}^{\infty} B_n(\lambda; x)\frac{t^n}{n!} = \frac{(\log \lambda + t)e^{tx}}{\lambda e^t - 1}.$$

By using Taylor expansion of e^{tx} in the left side of the above equation, after some elementary calculations, we obtain the desired result. □

We now give the distribution of the $\lambda-$Bernoulli polynomials.

Theorem 3. *Let $n \geq 0$, and let $d \in \mathbb{Z}^+$. Then we have*

$$B_n(\lambda; x) = d^{n-1} \sum_{a=0}^{d-1} \lambda^a B_n\left(\lambda^d; \frac{x + a}{d}\right). \tag{7}$$

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Proof. By using Eq-(6),

$$B_n(x; \lambda) = \int_{\mathbb{Z}_p} (x+y)^n \lambda^y d\mu_1(y)$$

$$= \lim_{N \to \infty} \frac{1}{dp^N} \sum_{y=0}^{dp^N-1} (x+y)^n \lambda^y$$

$$= \lim_{N \to \infty} \frac{1}{dp^N} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (a+dy+x)^n \lambda^{a+dy}$$

$$= d^{n-1} \lim_{N \to \infty} \frac{1}{p^N} \sum_{a=0}^{d-1} \lambda^a \sum_{y=0}^{p^N-1} \left(\frac{a+x}{d} + y\right)^n (\lambda^d)^y$$

$$= d^{n-1} \frac{1}{p^N} \sum_{a=0}^{d-1} \lambda^a \int_{\mathbb{Z}_p} \left(\frac{a+x}{d} + y\right)^n (\lambda^d)^y.$$

Thus, we have the desired result. □

By substituting $x = 0$ into Eq-(7), we have the following corollary:

Corollary 1. *For $m, n \in \mathbb{N}$, we have*

$$mB_n(\lambda) = \sum_{j=0}^{n} \binom{n}{j} B_j(\lambda^m) m^j \sum_{a=0}^{m-1} \lambda^a a^{n-j}. \tag{8}$$

(Gauss multiplicative formula for λ-Bernoulli numbers).

By Eq-(8), we have

Theorem 4. *For $m, n \in \mathbb{N}$ and $\lambda \in \mathbb{Z}_p$, we have*

$$mB_n(\lambda) - m^n[m]_\lambda B_n(\lambda^m) = \sum_{j=0}^{n-1} \binom{n}{j} B_j(\lambda^m) m^j \sum_{k=1}^{m-1} \lambda^k k^{n-j}. \tag{9}$$

Theorem 5. *Let $k \in \mathbb{Z}$, with $k > 1$. Then we have*

$$B_l(\lambda; k) - \lambda^{-k} B_l(\lambda) = \lambda^{-k} l \sum_{n=0}^{k-1} \lambda^n n^{l-1} + (\lambda^{-k} \log \lambda) \sum_{n=0}^{k-1} n^l \lambda^l. \tag{10}$$

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Proof. We set

$$-\sum_{n=0}^{\infty} e^{(n+k)t}\lambda^n + \sum_{n=0}^{\infty} e^{nt}\lambda^{n-k} = \sum_{n=0}^{\infty} e^{nt}\lambda^{n-k} \tag{10a}$$

$$= \sum_{l=0}^{\infty}(\lambda^{-k}\sum_{n=0}^{k-1} n^l\lambda^n)\frac{t^l}{l!}$$

$$= \sum_{l=1}^{\infty}(\lambda^{-k}l\sum_{n=0}^{k-1} n^{l-1}\lambda^n)\frac{t^{l-1}}{l!}.$$

Multiplying $(t + \log\lambda)$ both side of Eq-(10a), then by using Eq-(3) and Eq-(5), after some elementary calculations, we have

$$\sum_{l=0}^{\infty}(B_l(\lambda;k) - \lambda^{-k}B_l(\lambda))\frac{t^l}{l!}$$

$$= \sum_{l=0}^{\infty}(\lambda^{-k}l\sum_{n=0}^{k-1}\lambda^n n^{l-1} + \lambda^{-k}\log\lambda\sum_{n=0}^{k-1} n^l\lambda^l)\frac{t^l}{l!}. \tag{10b}$$

By comparing coefficient $\dfrac{t^l}{l!}$ in both sides of Eq-(10b). Thus we arrive at the Eq-(10). Thus we complete the proof of theorem. $\square$

Observe that $\lim_{\lambda\to1} B_l(\lambda) = B_l$. For $\lambda \to 1$, then Eq-(10) reduces the following:

$$B_l(k) - B_l = l\sum_{n=0}^{k-1} n^{l-1}.$$

If $\lambda \in T_p$, then Eq-(10) reduces to the following formula:

$$B_l(\lambda;k) - \lambda^{-k}B_l(\lambda) = \lambda^{-k}l\sum_{n=0}^{k-1}\lambda^n n^{l-1}.$$

Remark. Garrett and Hummel [1B] proved combinatorial proof of q-analogue of

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{k}^2$$

as follows:

$$\sum_{k=1}^{n} q^{k-1}[k]_q^2 \left(\begin{bmatrix} k-1 \\ 2 \end{bmatrix}_{q^2} + \begin{bmatrix} k+1 \\ 2 \end{bmatrix}_{q^2}\right) = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2,$$

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where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^{k} \dfrac{[n+1-j]_q}{[j]_q}$, q-binomial coefficients. Garrett and Hummel, in their paper, asked for a simpler q-analogue of the sums of cubes. As a response to Garrett and Hummel's question, in [11], Kim constructed the following formula

$$
S_{n,q^h}(k) = \sum_{l=0}^{k-1} q^{h^l}[l]^n
$$

$$
= \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \beta_{j,q} q^{kj}[k]^{n+1-j} - \frac{(1-q^{(n+1)k})\beta_{n+1,q}}{n+1},
$$

where $\beta_{j,q}$ are the q-Bernoulli numbers which were defined by

$$
e^{\frac{t}{1-q}} \frac{q-1}{\log q} - t \sum_{n=0}^{\infty} q^{n+x} e^{[n+x]t} = \sum_{n=0}^{\infty} \frac{\beta_{n,q}(x)}{n!} t^n, \quad |q| < 1, |t| < 1,
$$

$\beta_{n,q}(0) = \beta_{n,q}$. cf. [11]

Schlosser [20] gave for $n = 1, 2, 3, 4, 5$ the value of $S_{n,q^h}[k]$. In [28], the authors also gave another proof of $S_{n,q}(k)$ formula.

§2. Hurwitz's type λ-zeta function

In this section, by using generating function of λ-Bernoulli polynomials, we construct Hurwitz's type λ-zeta function, which is interpolate λ-Bernoulli polynomials at negative integers. By Eq-(5), we get

$$
F_\lambda(t; x) = \frac{\log \lambda + t}{\lambda e^t - 1} e^{xt} = -(\log \lambda + t) \sum_{n=0}^{\infty} \lambda^n e^{(n+x)t}
$$

$$
= \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}.
$$

By using $\dfrac{d^k}{dt^k}$ derivative operator to the above, we have

$$
B_k(\lambda; x) = \left. \frac{d^k}{dt^k} F_\lambda(t; x) \right|_{t=0},
$$

$$
B_k(\lambda; x) = -\log \lambda \sum_{n=0}^{\infty} \lambda^n (n+x)^k - k \sum_{n=0}^{\infty} (n+x)^{k-1} \lambda^n.
$$

Thus we arrive at the following theorem:

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Theorem 6. *For $k \geq 0$, we have*

$$-\frac{1}{k}B_k(\lambda; x) = \frac{\log \lambda^k}{k} \sum_{n=0}^{\infty} \lambda^n (n+x)^k + \sum_{n=0}^{\infty} \lambda^n (n+x)^{k-1}.$$

Consequently, we define Hurwitz type zeta function as follows:

Definition 1. Let $s \in \mathbb{C}$. Then we define

$$\zeta_\lambda(s, x) = \frac{\log \lambda}{1-s} \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+x)^{s-1}} + \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+x)^s}. \tag{11}$$

Note that $\zeta_\lambda(s, x)$ is analytic continuation, except for $s = 1$, in whole complex plane. By Definition 1 and Theorem 6, we have the following:

Theorem 7. *Let $s = 1 - k$, $k \in \mathbb{N}$,*

$$\zeta_\lambda(1 - k, x) = -\frac{B_k(\lambda, x)}{k}. \tag{12}$$

§3. Generalized λ-Bernoulli numbers associated with Dirichlet type λ-L-functions

By using Eq-(0), we define

$$I_1(f_d) = I_1(f) + \sum_{j=0}^{d-1} f'(j), \tag{12}$$

where $f_d(x) = f(x + d)$, $\int_{\mathbb{X}} f(x) d\mu(x) = I_1(f)$.

Let χ be Dirichlet character with conductor $d \in \mathbb{N}^+$, $\lambda \in \mathbb{Z}_p$.

By substituting $f(x) = \lambda^x \chi(x) e^{tx}$ into Eq-(12), then we have

$$\int_{\mathbb{X}} \chi(x) \lambda^x e^{tx} d\mu_1(x) = \sum_{j=0}^{d-1} \frac{\chi(j) \lambda^j e^{tj} (\log \lambda + t)}{\lambda^d e^{dt} - 1}$$

$$= \sum_{n=0}^{\infty} B_{n,\chi}(\lambda) \frac{t^n}{n!}. \tag{12a}$$

By Eq-(12a), we easily see that

$$B_{n,\chi}(\lambda) = \int_{\mathbb{X}} \chi(x) x^n \lambda^x d\mu_1(x). \tag{12b}$$

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From Eq-(12a), we define generating function of generalized Bernoulli number by

$$F_{\lambda,\chi}(t) = \sum_{j=0}^{d-1} \frac{\chi(j)\lambda^j e^{tj}(\log \lambda + t)}{\lambda^d e^{dt} - 1} = \sum_{n=0}^{\infty} B_n(\lambda)\frac{t^n}{n!}. \tag{12c}$$

Observe that if $\lambda \in T_p$, then the above formula reduces to

$$F_{\lambda,\chi}(t) = \sum_{j=0}^{d-1} \frac{\chi(j)\lambda^j e^{tj}t}{\lambda^d e^{dt} - 1} = \sum_{j=0}^{\infty} B_n(\lambda)\frac{t^n}{n!} \quad (\text{ for detail see cf. } [3,16,18,22,23,24]).$$

From the above, we easily see that

$$F_{\lambda,\chi}(t) = -(\log \lambda + t)\sum_{m=1}^{\infty} \chi(m)\lambda^m e^{tm} = \sum_{n=0}^{\infty} B_{n,\chi}(\lambda)\frac{t^n}{n!}.$$

By applying $\left.\dfrac{d^k}{dt^k}\right|_{t=0}$ derivative operator both sides of the above equation, we arrive at the following theorem:

Theorem 8. *Let $k \in \mathbb{Z}^+$, $\lambda \in \mathbb{Z}_p$ and let χ be Derichlet character with conductor d. Then we have*

$$\sum_{m=1}^{\infty} \chi(m)\lambda^m m^{k-1} + \frac{\log \lambda}{k} \sum_{m=1}^{\infty} \lambda^m \chi(m)m^k = -\frac{B_{k,\chi}(\lambda)}{k}. \tag{13}$$

Definition 2(Dirichlet type λ-L function). For $\lambda, s \in \mathbb{C}$, we define

$$L_\lambda(s,\chi) = \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^s} - \frac{\log \lambda}{s-1} \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^{s-1}}. \tag{14}$$

Relation between $L_\lambda(s,\chi)$ and $\zeta_\lambda(s,y)$ is given by the following theorem :

Theorem 9. *Let $s \in \mathbb{C}$ and $d \in \mathbb{Z}^+$. Then we have*

$$L_\lambda(s,\chi) = d^{-s}\sum_{a=1}^{d} \lambda^a \chi(a)\zeta_{\lambda^d}\left(s,\frac{a}{d}\right).$$

Proof. By substituting $m = a + dk$, $a = 1, 2, \cdots, d$, $k = 0, 1, \cdots, \infty$, into Eq-(14), we have

$$L_\lambda(s,\chi) = \sum_{a=1}^{d}\sum_{k=0}^{\infty} \frac{\lambda^{a+dk}\chi(a+dk)}{(a+dk)^s} - \frac{\log \lambda}{s-1}\sum_{a=1}^{d}\sum_{k=0}^{\infty} \frac{\lambda^{a+dk}\chi(a+dk)}{(a+dk)^{s-1}}$$

$$= d^{-s}\sum_{a=1}^{d}(\lambda^a \chi(a))\left[\sum_{k=0}^{\infty} \frac{(\lambda^d)^k}{(k+\frac{a}{d})^s} - \frac{\log \lambda^d}{s-1}\sum_{k=0}^{\infty} \frac{(\lambda^d)^k}{(k+\frac{a}{d})^{s-1}}\right].$$

By using Eq-(11) in the above we obtain the desired result. $\square$

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Theorem 10. *For $k \in \mathbb{Z}^+$, we have*

$$L_\lambda(1 - k, \chi) = -\frac{1}{k}B_{k,\chi}(\lambda), \quad k > 0.$$

Proof. By substituting $s = 1 - k$ in Definition 2 and using Eq-(13), we easily obtain the desired result. $\square$

Remark. If $\lambda \in T_p$, then from Definition 2, we have

$$L_\lambda(s, \chi) = \sum_{m=1}^{\infty} \frac{\lambda^m \chi(m)}{m^s}.$$

In [18], Koblitz studied on this function. He gave the name of this function "twisted L-function" for λ is r-th root of 1. In [22,23,24], Simsek studied of this functions. He gave fundamental properties of this function as well.

In [16], Kim et all gave λ-(h, q) zeta function and λ-(h, q) L-function. These functions interpolate $\lambda - (h, q)$-Bernoulli numbers. Observe that, if we take $s = 1 - k$ in Theorem 9, and then using Eq-(12) in Theorem 7, we get another proof of Theorem 10.

§4. λ-Bernoulli numbers of order r associated with multiple zeta function

In this section, we define generating function of λ-Bernoulli numbers of order r. By using Mellin transforms and Cauchy residue theorem, we obtain multiple zeta function which is given in Eq-(C). We also gave relations between λ-Bernoulli polynomials of order r and multiple zeta function at negative integers. This relation is important and very interesting. Let $r \in \mathbb{Z}^+$. Generating function of λ-Bernoulli numbers of order r is defined by

$$F_\lambda^{(r)}(t) = \left(\frac{\log \lambda + t}{\lambda e^t - 1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda)\frac{t^n}{n!}. \tag{15}$$

Generating function of λ-Bernoulli polynomials of order r is defined by

$$F_\lambda^{(r)}(t, x) = F_\lambda^{(r)}(t)e^{tx} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda)\frac{t^n}{n!}.$$

Observe that when $r = 1$, Eq-(15) reduces to Eq-(3). By applying Mellin transforms to the Eq-(15) we get

$$\frac{1}{\Gamma(s)}\int_0^{\infty} \lambda^r e^{-tr}F_\lambda^{(r)}(-t)(t - \log \lambda)^{s-r-1}dt = \sum_{n_1,\cdots,n_r=0}^{\infty} \frac{1}{(n_1 + n_2 + \cdots + n_r + r)^s}.$$

Thus, we get, by (C)

$$\zeta_r(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} \lambda^r e^{-tr}F_\lambda^{(r)}(-t)(t - \log \lambda)^{s-r-1}dt.$$

By using the above relation, we obtain the following theorem:

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Theorem 11. *Let $r, m \in \mathbb{Z}^+$. Then we have*

$$\zeta_r(-m) = (-\lambda)^r m! \sum_{j=0}^{\infty} \binom{-m-r-1}{j} (\log \lambda)^j \frac{B_{m+r+j}(\lambda; r)}{(m+r+j)!}. \qquad (D1)$$

Remark. If $\lambda \to 1$, the above theorem reduces to

$$\zeta_r(-m) = (-1)^r m! \frac{B_{m+r}(1; r)}{(m+r)!} \qquad (D2)$$

which is given Theorem 6 in [12].

By (D1) and (D2), we obtain relation between λ-Bernoulli polynomials of order r and ordinary Bernoulli polynomials of order r as follows:

$$B_{m+r}(r) = \lambda^r \sum_{j=0}^{\infty} \binom{-m-r-1}{j} (\log \lambda)^j \frac{B_{m+r+j}(\lambda; r)}{(m+r+j)!} (m+r)!$$

where $m, r \in \mathbb{Z}^+$.

We now give relations between $B_n^{(r)}(\lambda)$ and $H_n^{(r)}(\lambda^{-1})$ as follows:
If $\lambda \in T_p$, then Eq-(15) reduces to the following equation

$$\frac{t^r}{(\lambda e^t - 1)^r} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!}.$$

Thus by the above equation, we easily see that

$$t^r = (\lambda e^t - 1)^r e^{B^{(r)}(\lambda)t}$$

$$= \sum_{l=0}^{r} \lambda^l (-1)^{r-l} e^{(B^{(r)}(\lambda) + l)t}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{r} \lambda^l (-1)^{r-l} (B^{(r)}(\lambda) + l)^n \right) \frac{t^n}{n!}.$$

Consequently we have

$$\sum_{l=0}^{r} \lambda^l (-1)^{r-l} (B^{(r)}(\lambda) + l)^n = \begin{cases} 0 & \text{if } n \neq r \\ 1 & \text{if } n = r. \end{cases}$$

By Eq-(15) we obtain

$$\sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!} = \frac{t^r}{(\lambda - 1)^r} \sum_{n=0}^{\infty} H_n^{(r)}(\lambda^{-1}) \frac{t^n}{n!}.$$

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By comparing coefficient $\dfrac{t^n}{n!}$ in the both sides of the above equation, we have

$$B_{n+r}^{(r)}(\lambda) = \frac{\Gamma(n+r+1)}{\Gamma(n+1)} \frac{1}{(\lambda-1)^r} H_n^{(r)}(\lambda^{-1}).$$

Observe that, if we take $r = 1$, then the above identity reduce to Eq-(4), that is

$$B_{n+1}(\lambda) = \frac{(n+1)}{\lambda-1} H_n(\lambda^{-1}).$$

§5. λ-Bernoulli numbers and polynomials associated with multivariate p-adic invariant integral

In this section, we give generalized λ-Bernoulli numbers of order r. Consider the multivariate p-adic invariant integral on $\mathbb{Z}_p$ to define λ-Bernoulli numbers and polynomials.

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r-\text{times}} \lambda^{w_1 x_1 + \cdots + w_r x_r} e^{(w_1 x_1 + \cdots + w_r x_r)t} d\mu_1(x_1) \cdots d\mu_1(x_r)$$

$$= \frac{(w_1 \log \lambda + w_1 t) \cdots (w_r \log \lambda + w_r t)}{(\lambda^{w_1} e^{w_1 t} - 1) \cdots (\lambda^{w_r} e^{w_r t} - 1)} \qquad (16)$$

$$= \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; w_1, w_2, \cdots, w_r) \frac{t^n}{n!},$$

where we called $B_n^{(r)}(\lambda; w_1, w_2, \cdots, w_r)$ λ-extension of Bernoulli numbers. Substituting $\lambda = 1$ into Eq-(16), λ-extension of Bernoulli numbers reduce to Barnes Bernoulli numbers as follows :

$$\frac{(w_1 t) \cdots (w_r t)}{(e^{w_1 t} - 1) \cdots (e^{w_r t} - 1)} = \sum_{n=0}^{\infty} B_n^{(r)}(w_1, \cdots, w_r) \frac{t^n}{n!},$$

where $B_n^{(r)}(w_1, \cdots, w_r)$ are denoted Barnes Bernoulli umbers and $w_1, \cdots, w_r$ complex numbers with positive real parts [1A,7,27]. Observe that when $w_1 = w_2 = \cdots = w_r = 1$ in Eq-(16), we obtain the λ-Bernoulli numbers of higher order as follows:

$$\left(\frac{\log \lambda + t}{\lambda e^t - 1} \right)^r = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda) \frac{t^n}{n!}.$$

We note that $B_n^{(r)}(\lambda; 1, 1, \cdots, 1) = B_n^{(r)}(\lambda)$.

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Consider

$$\left(\frac{\log \lambda + t}{\lambda e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x)\frac{t^n}{n!}.$$

Observe that

$$\sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x)\frac{t^n}{n!} = \left(\frac{\log \lambda + t}{\lambda e^t - 1}\right)^r e^{(\log \lambda + t)x}\lambda^{-x}$$

$$= \frac{1}{\lambda^x} \sum_{m=0}^{\infty} B_m^{(r)}(\lambda; x)\frac{(t + \log \lambda)^m}{m!}$$

$$= \frac{1}{\lambda^x} \sum_{m=0}^{\infty} \frac{B_m^{(r)}(\lambda; x)}{m!} \sum_{l=0}^{m} \binom{m}{l}(\log \lambda)^m t^{m-l}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{\lambda^x} \sum_{l=0}^{\infty} \frac{B_{n+l}^{(r)}(\lambda; x)}{l!}(\log \lambda)^l\right)\frac{t^n}{n!}.$$

Now, comparing coefficient $\dfrac{t^n}{n!}$ both sides of the above equation, we easily arrive at the following theorem:

Theorem 12. *For $n, r \in \mathbb{N}$ and $\lambda \in \mathbb{Z}_p$, we have*

$$B_n^{(r)}(\lambda; x) = \frac{1}{\lambda^r} \sum_{l=0}^{\infty} B_{n+l}^{(r)}(\lambda; x)\frac{(\log \lambda)^l}{l!},$$

where $0^l = \begin{cases} 1 & if\ l = 0 \\ 0 & if\ \ l \neq 0 \end{cases}.$

Remark. In Theorem 12, we see that

$$\lim_{\lambda \to 1} B_n^{(r)}(\lambda; x) = \begin{cases} B_n^{(r)}(x) & \text{if } l = 0, \\ 0 & \text{if } \ l \neq 0. \end{cases}$$

§6. λ-Bernoulli numbers and polynomials in the space of locally constant

In this section, we construct partial λ-zeta functions, we need this function in the following section. We need this function in the following section. By Eq-(3b), Frobenius-Euler polynomials are defined by means of the following generating function:

$$\left(\frac{1 - u}{e^t - u}\right) e^{xt} = \sum_{n=0}^{\infty} H_n(u, x)\frac{t^n}{n!}.$$

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As well known, we note that the Frobenius-Euler polynomials of order r were defined by

$$\left(\frac{1-u}{e^t - u}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(u, x)\frac{t^n}{n!}.$$

The case $x = 0$, $H_n^{(r)}(u, 0) = H_n^{(r)}(u)$, which are called Frobenius-Euler numbers of order r.

If $\lambda \in T_p$, then λ-Bernoulli polynomials of order r are given by

$$\frac{t^r}{(\lambda e^t - 1)^r}e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(\lambda; x)\frac{t^n}{n!}.$$

Hurwitz type λ-zeta function is given by

$$\zeta_\lambda(s, x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(n + x)^s}, \qquad \lambda \in T_p. \tag{17}$$

Thus, from Theorem 7, we have

$$\zeta_\lambda(1 - k, x) = -\frac{1}{k}B(\lambda; x), \qquad k \in \mathbb{Z}^+. \tag{17a}$$

We now define λ-partial zeta function as follows

$$H_\lambda(s, a|F) = \sum_{m \equiv a \ (\text{mod } F)} \frac{\lambda^m}{m^s}. \tag{17b}$$

From (17), we have

$$H_\lambda(s, a|F) = \frac{\lambda^a}{F^s}\zeta_{\lambda^F}\left(s, \frac{a}{F}\right), \tag{17c}$$

where $\zeta_{\lambda^F}\left(s, \frac{a}{F}\right)$ is given by Eq-(17). By Eq-(17a) we have

$$H_\lambda(1 - n, a|F) = -\frac{F^{n-1}\lambda^a B_n(\lambda^F; \frac{a}{F})}{n}, \qquad n \in \mathbb{Z}^+. \tag{18}$$

If $\lambda \in T_p$, then by Eq-(14), we have

$$L_\lambda(s, \chi) = \sum_{n=1}^{\infty} \frac{\lambda^n \chi(n)}{n^s},$$

where $s \in \mathbb{C}$, χ be the primitive Dirichlet character with conductor $f \in \mathbb{Z}^+$. By Theorem 9, Eq-(17c) and Eq-(18) we easily see that

$$L_\lambda(s, \chi) = \sum_{a=1}^{F} \chi(a) H_\lambda\left(s, \frac{a}{F}\right),$$

and

$$L_\lambda(1 - k, \chi) = -\frac{B_{k,\chi}(\lambda)}{k}, \quad k \in \mathbb{Z}^+,$$

where $B_{k,\chi}(\lambda)$ is defined by

$$\sum_{a=0}^{F-1} \frac{t\lambda^a \chi(a) e^{at}}{\lambda^F e^{Ft} - 1} = \sum_{a=0}^{\infty} B_{n,\chi}(\lambda) \frac{t^n}{n!}, \quad \lambda \in T_p$$

and F is multiple of f.

Remark.

$$\frac{B_m(\lambda)}{m} = \frac{1}{\lambda - 1} H_{n-1}(\lambda^{-1}), \quad \lambda \in T_p.$$

§7. p-adic interpolation function

In this section we give p-adic λ-L function. Let w be the Teichimuller character and let $< x >= \dfrac{x}{w(x)}$.

When F is multiple of p and f and $(a, p) = 1$, we define

$$H_{p,\lambda}(s, a|F) = \frac{1}{s-1} \lambda^a < a >^{1-s} \sum_{j=0}^{\infty} \binom{1-s}{j} \left(\frac{F}{a}\right)^j B_j(\lambda^F).$$

From this we note that

$$H_{p,\lambda}(1 - n, a|F) = -\frac{1}{n}\frac{\lambda^a}{F} < a >^n \sum_{j=0}^{n} \binom{n}{j} \left(\frac{F}{a}\right)^j B_j(\lambda^F)$$

$$= -\frac{1}{n} F^{n-1} \lambda^a w^{-n}(a) B_n(\lambda^F; \frac{a}{F})$$

$$= w^{-n}(a) H_\lambda(1 - n; \frac{a}{F}),$$

since by Theorem 3 for $\lambda \in T_p$, Eq-(18).

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By using this formula, we can consider p-adic λ-L-function for λ-Bernoulli numbers as follows:

$$L_{p,\lambda}(s, \chi) = \sum_{\substack{a=1 \\ p \nmid a}}^{F} \chi(a) H_{p,\lambda}\left(s, \frac{a}{F}\right).$$

By using the above definition, we have

$$L_{p,\lambda}(1-n, \chi) = \sum_{\substack{a=1 \\ p \nmid a}}^{F} \chi(a) H_{p,\lambda}\left(1-n, \frac{a}{F}\right)$$

$$= -\frac{1}{n}\left(B_{n,\chi w^{-n}}(\lambda) - p^{n-1}\chi w^{-n}(p) B_{n,\chi w^{-n}}(\lambda^p)\right).$$

Thus, we define the formula

$$L_{p,\lambda}(s, \chi) = \frac{1}{F}\frac{1}{s-1}\sum_{a=1}^{F} \chi(a)\lambda^a <a>^{1-s} \sum_{j=0}^{\infty}\binom{1-s}{j} B_j(\lambda^F)$$

for $s \in \mathbb{Z}_p$.

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